

A knot bounding a grope of class n is $\lceil \frac{n}{2} \rceil$ -trivial *

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Abstract

In this article it is proven that if a knot, K , bounds an imbedded grope of class n , then the knot is $\lceil \frac{n}{2} \rceil$ -trivial in the sense of Gusarov and Stanford. That is, all type $\lceil \frac{n}{2} \rceil$ invariants vanish on K . We also give a simple way to construct all knots bounding a grope of a given class. It is further shown that this result is optimal in the sense that for any n there exist gropes which are not $\lceil \frac{n}{2} \rceil + 1$ -trivial.

1 Introduction

1.1 Origins

Finite type invariants have been a hot topic of study in recent years, having first been introduced in proto-form in a seminal paper of Vassiliev[V], from which derives their alternative moniker “Vassiliev invariants”. Birman and Lin [BL], upon reading Vassiliev’s paper were able to give the by now familiar simple axiomatic condition for being a finite type invariant of type n : Given a knot invariant ν taking values in an abelian group extend it to knots with finitely many transverse double points by the following formula, obligatory in any paper on finite type invariants.

$$\nu(\text{X}) = \nu(\text{X}) - \nu(\text{X})$$

The invariant ν is *finite type of type n* iff it vanishes on knots with $n + 1$ double points. Birman and Lin also proved that the coefficients of x^n in the Jones polynomial under the change of variables $t \rightarrow e^x$ are type n invariants. This is actually equivalent to saying that the n th derivatives $J^{(n)}(1)$ are type n invariants, and indeed this is used in the last section of the present paper.

1.2 The work of Gusarov

Gusarov[G] takes a different tack, constructing a group of knots, \mathcal{G}_n , which is a quotient of the monoid of knots under connected sum. The equivalence relation, as proven by

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Stanford and Ng[NS], may be chosen to be that two knots are equivalent iff all additive type n invariants are the same. An alternate description of this group is given as follows. Given a knot K choose $n + 1$ disjoint groups of crossing changes $S = \{s_1, \dots, s_{n+1}\}$ for the knot. (S is called a *scheme* by Gusarov, or at least by his translator.) If this scheme has the property that for some L , $K_\sigma = L$ (K_σ is the knot modified along the crossing changes in σ .) for all nonempty σ , then we say $K \sim_n L$. (K is n -equivalent to L .)

Remarkably, \sim_n is an equivalence relation [NS]. If we quotient the monoid of knots under $\#$ by \sim_n we recover Gusarov's group \mathcal{G}_n . Denote elements in \mathcal{G}_n by $[K]_n$, where K is a knot representing the equivalence class $[K]_n$. In fact, for any scheme S the element $Tot(K; S) \in \mathcal{G}_n$ is trivial where $Tot(K; S) := \sum_{\sigma \subset S} (-1)^{|\sigma|} [K_\sigma]_n$, where $|\sigma|$ is the cardinality of σ . Indeed this is the main tool of the present paper. This expands on the idea of Lin and Kalfagianni [L-K] to just use the relation \sim_n . Also, if we extend the above definition of finite type invariants to links, this formula still holds in the following sense. Let μ be a type n invariant and S a scheme of $n + 1$ sets of crossing changes of a link L , then

$$\sum_{\sigma \subset S} (-1)^{|\sigma|} \mu(L_\sigma) = 0. \quad (1)$$

[Proof] (Following [G], Lemma 5.2)

An immediate consequence of the finite type axiom is the following:

If S is a scheme of cardinality $n + 1$ on L where each $s_i \in S$ is a single crossing change, then $\sum_{\sigma \subset S} (-1)^{|\sigma|} \mu(L_\sigma) = 0$, if μ is a type n invariant. Our task is to prove this when the s_i contain more than one crossing change. We induct on say $\sum_{i=1}^{n+1} |s_i|$. Given S where $\sum |s_i| > n + 1$, and suppose without loss of generality that $s_1 = s'_1 \cup s''_1$ is a partition of s_1 into two nonempty sets. We define 2 schemes of lower complexity:

$$\begin{aligned} S' &= \{s'_1, s_2, \dots, s_{n+1}\} \text{ on } L \\ S'' &= \{\hat{s}_1'', \hat{s}_2, \dots, \hat{s}_{n+1}\} \text{ on } L_{s'_1} \end{aligned}$$

where, if s is a move on L , the move \hat{s} denotes the induced move on the link modified along s'_1 .

$$\begin{aligned} 0 + 0 &= \sum_{\sigma \subset S'} (-1)^{|\sigma|} \mu(L_\sigma) + \sum_{\sigma \subset S''} (-1)^{|\sigma|} \mu((L_{s'_1})_\sigma) = \\ &= \left(\sum_{\substack{\sigma \subset S' \\ s'_1 \in \sigma}} (-1)^{|\sigma|} \mu(L_\sigma) + \sum_{\substack{\sigma \subset S'' \\ s'_1 \notin \sigma}} (-1)^{|\sigma|} \mu((L_{s'_1})_\sigma) \right) + \\ &\quad \sum_{\substack{\sigma \subset S' \\ s'_1 \notin \sigma}} (-1)^{|\sigma|} \mu(L_\sigma) + \sum_{\substack{\sigma \subset S'' \\ s'_1 \in \sigma}} (-1)^{|\sigma|} \mu((L_{s'_1})_\sigma) \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{\substack{\sigma \subset S' \\ s'_1 \in \sigma}} (-1)^{|\sigma|} \mu(L_\sigma) + \sum_{\substack{\sigma \subset S' \\ s'_1 \in \sigma}} (-1)^{|\sigma|+1} \mu((L_{s'_1})_\sigma) \right) + \\
&\quad \sum_{\substack{\sigma \subset S \\ s_1 \notin \sigma}} (-1)^{|\sigma|} \mu(L_\sigma) + \sum_{\substack{\sigma \subset S \\ s_1 \in \sigma}} (-1)^{|\sigma|} \mu((L_{s_1})_\sigma) \\
&= 0 + \sum_{\sigma \subset S} (-1)^{|\sigma|} \mu(L_\sigma) \square
\end{aligned}$$

Note that this lemma generalizes the fact that $Tot(K; S) = 0$ in two senses: a) it holds for non-additive knot invariants and b) it holds for *link* invariants.

We'd like to point out also that a “set of crossing changes” s_i can be thought of a homotopy of the knot (or link) supported in a disjoint union of balls. Indeed it is useful to think of it this way, in which case a scheme S is a set of “disjointly supported homotopies.” (Any homotopy of a knot beginning and ending with an embedding is equivalent to a homotopy which is a set of disjointly supported finger moves, i.e. crossing changes.)

1.3 Gropes

A grope, G , of class n , loosely, is a 2-complex representing an n commutator [FT]. To define gropes recursively, however, we use a different quantity, *depth*. A depth 1 grope is defined to be a circle, while a depth 2 grope is defined as a punctured surface. If you know what a depth $< n$ grope is, to form a grope, G , of depth n , you take a punctured surface and to each element of a prescribed symplectic basis you glue a grope with depth $< n$, such that at least one of these attached gropes is of depth $n - 1$.

The class of a grope G is the term of the lower central series that the boundary circle represents, or explicitly, if $\{\alpha_i, \beta_i\}$ is the symplectic basis and A_i, B_i are the gropes to be added, then $\text{class}(G) = \min_i \{\text{class}(A_i) + \text{class}(B_i)\}$.

1.4 Incorporating some geometry

My result is then that if a knot bounds a grope of class n , imbedded in \mathbb{R}^3 , that that knot is trivial in $\mathcal{G}_{\lceil \frac{n}{2} \rceil}$. To do this I make repeated use of the fact that all the $Tot(K; S)$'s are trivial in the group $\mathcal{G}_{|S|-1}$ by finding appropriate collections of disjointly supported homotopies, the most interesting of which come from the in/out trick defined in section 4. In a sense the main theorem is pretty easy to prove if you don't mind not getting the optimal result. That is, without the in/out trick, it is not so hard to prove that class n gropes are $\lfloor \frac{n}{2} \rfloor - 1$ -trivial. It is the in/out trick which allows one to get those two extra groups of crossing changes for odd n .

It was originally suggested by Mike Freedman that class n gropes might always be $n - 1$ trivial, (e.g. one group of crossing changes for every ‘tip’.) This turns out to be overambitious by a factor of 2 and in the last section we indeed deduce the existence of class n gropes that are not $\lceil \frac{n}{2} \rceil + 1$ -trivial.

An interesting consequence of the main theorem, (or the slightly weaker one mentioned above) is that a knot bounding a grope of arbitrarily large class cannot be distinguished from the unknot by finite type invariants. It is a conjecture of Mike Freedman's that this phenomenon is impossible. Indeed he conjectures that in any three manifold, you cannot have an infinite imbedded grope, every stage of which is incompressible.

1.5 The work of X.S. Lin and E. Kalfagianni

The main theorem of this paper is similar to that of a paper of X.S. Lin and E. Kalfagianni[LK]. In that paper it is proven that knots which bound certain immersed gropes of height $n + 2$ are $l(n)$ -trivial, where $\lim_{n \rightarrow \infty} l(n) = \infty$. More specifically, they consider immersed gropes such that all self-intersections occur away from the bottom stage. There is also the restriction that the bottom stage is *regular*, which among other things implies that the complement of the Seifert surface has free fundamental group. (It should be noted that the obvious generalization of their and my result, that all knots bounding immersed gropes are to some degree trivial, meets with the problem that all knots bound immersed gropes of arbitrary height, since the lower central series of a knot complement stabilizes at $\Gamma^2 = \mathbb{Z}$.) The method of proof of their theorem, which precedes mine, is to find crossing changes which implement the group-theoretic $n - 1$ -triviality of an n -commutator, and as such is similar to the ideas presented in the present paper, although the actual geometric implementation is quite different. For instance their paper is mainly concerned with the analysis of planar combinatorics, whereas mine uses more three dimensional arguments.

It turns out in their case, as well as mine, that one can not fully realize the degree of triviality present in an n -commutator, the problem being that one must be able to find n geometric independent moves which have the effect of deleting a letter in the commutator. For the case of imbedded gropes one can show that only $\lceil \frac{n}{2} \rceil + 1$ of the moves are actually realizable independently, whereas in the case of regular Seifert surfaces only $\log_4 n/3 + 1$ of the moves are realizable independently.

1.6 Synopsis

In section 1 the introduction was given.

In section 2 we figure out how to put the grope G into a nice form, and using this form, to associate a decorated graph $\Gamma(G)$ to G .

In section 3 we reduce to the case when the bottom stage(The bottom stage is the one whose boundary is the knot itself.) is genus 1, introducing two of the three types of moves (homotopies) we will need for the main theorem. We need to reduce to the genus 1 case in order to apply the in/out trick which only works for gropes with bottom stage genus 1.

In section 4 we describe the in/out trick, and give some applications. The trick is used in the proof of the main theorem and also in the construction of the knots in section 6.

In section 5, we finally polish off the main theorem.

In section 6, we show that our result is optimal.

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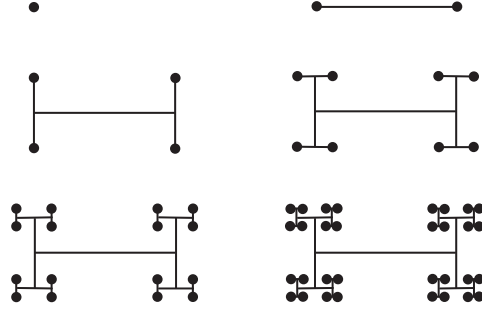


Figure 1: The 1-complexes Ξ_0 to Ξ_5

2 Standard position and the decorated graph $\Gamma(G)$

2.1 Standard position

We begin by finding a nice handlebody surrounding the grope. We need the following definition of a particular 1 complex.

Definition 2.1 *The 1-complex Ξ_i is defined for all $i \in \mathbb{N}$ inductively as follows. Ξ_0 is a point, while Ξ_1 is an interval. Now suppose Ξ_{i-1} is defined and has 1-valent vertices z_1, \dots, z_k . Form Ξ_i as the adjunction space gotten by gluing the midpoint of each of k intervals I_1, \dots, I_k to the corresponding z_1, \dots, z_k .*

Let v_i be imbedded circles representing the tips of the grope. For instance, if G is a genus 2 surface, there will be four v_i .

Theorem 2.1 *For every imbedded grope $G \subset \mathbb{R}^3$, there is a ball B and handles $H_i \cong D^2 \times I$ such that for all $t \in I$, the cross section $G \cap (D^2 \times \{t\}) \subset H_i$ is equal to $\Xi_{l(i)}$ for some $l(i)$. Further $v_i \cap (D^2 \times \{t\})$ is just a point in $\Xi_1 \subset \Xi_{l(i)}$. Also $H_i \cap B = D^2 \times \partial I$. We also want there to be disks $D_i \subset B$ where $\partial D_i = \gamma_i \cup \eta_i$ with $\gamma_i \cap \eta_i = \partial \eta_i = \partial \gamma_i$ such that $\gamma_i \subset v_i$ and $\eta_i \subset \partial B$ and such that $D_i \cap G = \gamma_i$. Also $\text{int} D_i \cap \text{int} D_j = \emptyset$ if $i \neq j$. Finally, we require that $B \cup_i H_i$ is a regular neighborhood of G .*

[Proof]

First we show this is true for some model \mathcal{G} of G in \mathbb{R}^3 . Once we do this, we are done. For if $f: \mathcal{G} \rightarrow G$ then f extends to give a PL-homeomorphism (or diffeomorphism depending on which category you prefer) of regular neighborhoods $\nu(\mathcal{G}) \rightarrow \nu(G)$, which will transport the structure given on the model. A grope G of depth 1 is just a circle. In this case we can let \mathcal{G} be an unknot. Take B to be a small neighborhood of some point of \mathcal{G} and take the single handle H_1 to be a regular neighborhood of the arc of \mathcal{G} outside of B . The disk D_1 is just a spanning disk of \mathcal{G} intersected with B .

Now, for the inductive step suppose we have a grope \mathcal{G} which is formed as follows. Suppose the genus of the bottom stage of \mathcal{G} is g with a symplectic basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ where G is formed by attaching gropes A_i, B_i to α_i, β_i respectively. Since A_i and B_i have

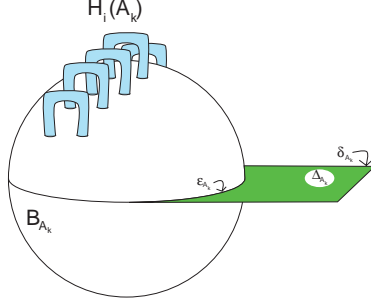


Figure 2:

lower depth than \mathcal{G} , the theorem is assumed to hold for them. So we have balls B_{A_i}, B_{B_i} together with handles $H_i(A_k), H_i(B_k)$ satisfying the hypotheses of the theorem. Let δ_{A_k} be a small subarc of ∂A_k inside B_{A_k} and ϵ_{A_k} a small arc joining the endpoints of δ_{A_k} but with $\text{int}(\epsilon_{A_k})$ contained in the interior of the bottom stage of A_k , such that $\delta_{A_k} \cup \epsilon_{A_k}$ bound a disk Δ_{A_k} . Modify B_{A_k} slightly, via a finger move disjoint from the rest of \mathcal{G} and the various D_i , so that $\partial(B_{A_k} \cup H_i(A_k)) \cap A_k = \epsilon_{A_k}$ with δ_{A_k} and Δ_{A_k} lying outside B_{A_k} . Do this similarly for B_k . See figure 2.

Now to form \mathcal{G} attach annuli to the A_k, B_k by gluing the cores of the annuli to the boundaries of A_k, B_k orthogonally to A_k, B_k . Modify these by plumbing together the A_k, B_k annuli for all k and then connect summing all these together as in figure 3, to form the genus g bottom stage of the grope. Our new handlebody $B \cup_i H_i$ is formed as pictured in figure 3. The new handles are the same as the old, but B is formed by taking a small regular neighborhood of $G \setminus (\bigcup_i H_i \cup B_{A_k} \cup B_{B_k})$.

If a handle looked like $\Xi_i \times I$ in A_k then in G it looks like $\Xi_{i+1} \times I$, the effect of attaching an annulus. Also, the v_i for G are just made of all the v_i for the A_k and B_k so they still lie nicely in the handles as a subset of $\Xi_1 \times I \subset \Xi_{i+1} \times I$. As for the existence of disks D_i , consider a handle H for G with tip v_H . By hypothesis, there is a disk $D_H \subset B_{A_k}$ which extends from v_H to ∂B_{A_k} and hence to ∂B . \square

This theorem gives a simple way of forming knots which bound gropes, since we can imbed the handlebody in any way we please in \mathbb{R}^3 . Notice that the cores, v_i , can be, using the disks D_i , extended disjointly along annuli to curves \bar{v}_i on $\partial(B \cup H_i)$. Now, proceeding with the advertised construction of the graph $\Gamma(G)$, we wish to group the cores v_i into collections of cores $V_i, i = 1, \dots, n$ where n is the class of G . We want these V_i to have the property that if the collection of cores in some V_i all bound disks, Δ_{ij} , into the complement of the grope, then the knot ∂G is isotopic to the unknot in a small regular neighborhood of $G \cup_{i,j} \Delta_{ij}$. We do this inductively as follows. For a grope with $k(G) = 2$, a Seifert surface, let V_1 be formed by choosing one v_i from each pair of dual bands. V_2 is the set containing all the other v_i . These obviously have the required property, since if V_1 bounds disks into the grope complement, surgery on these compressing disks gives a spanning disk of ∂G . Now a grope with $k(G) > 2$, is formed by gluing gropes of lower depth, say A_i, B_i to a symplectic basis of the bottom stage, α_i, β_i . Suppose the class of

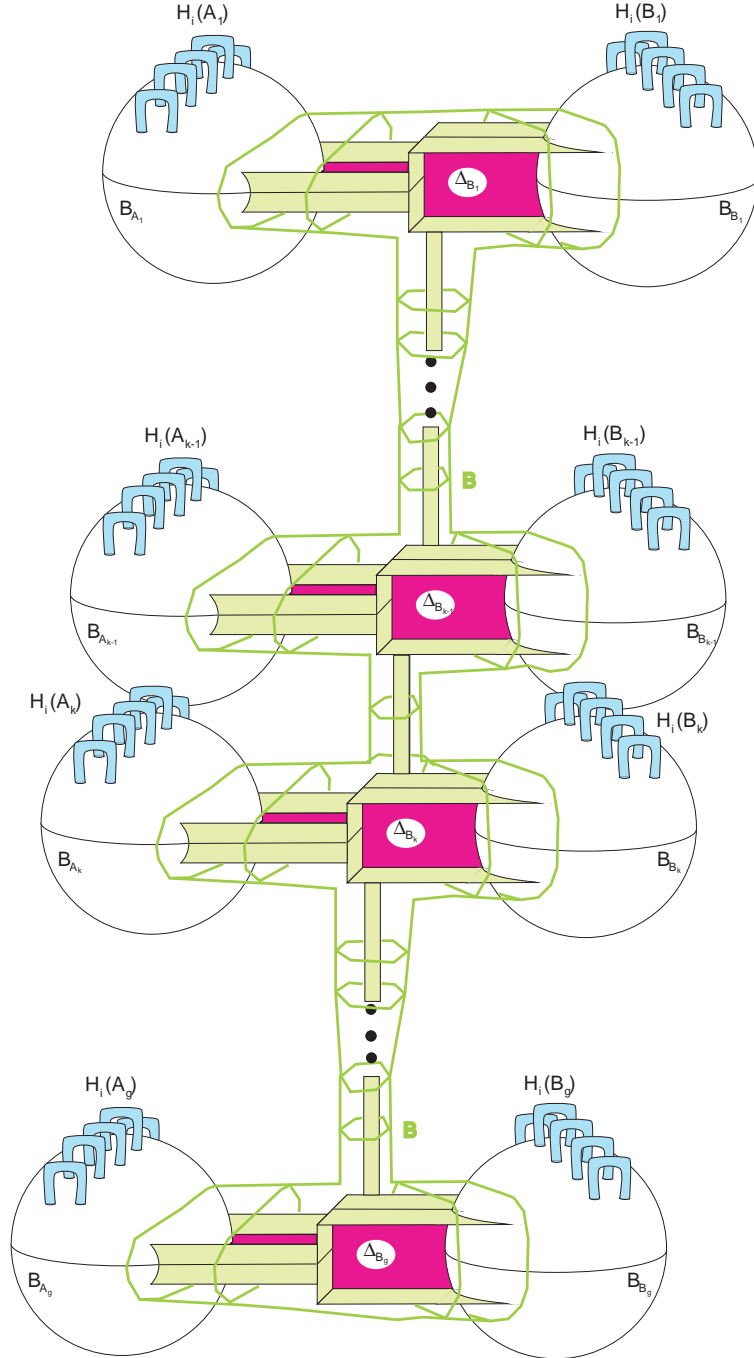


Figure 3:

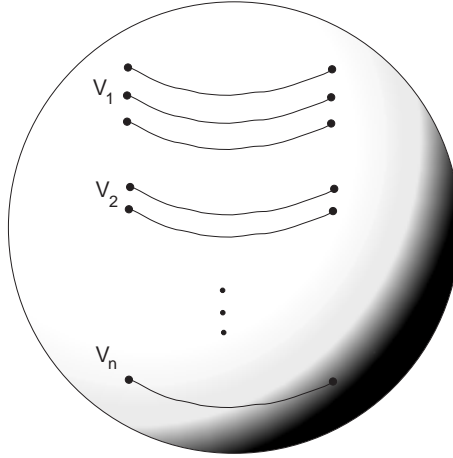


Figure 4:

A_i is s_i and that of B_i is t_i . Then by the inductive hypothesis we have groupings $V_j^{\alpha_i}$, $j = 1, \dots, s_i$, and $V_j^{\beta_i}$, $j = 1, \dots, t_i$. By definition the class of $G = \min_i \{s_i + t_i\} = n$, say. Let the elements of $\{V_j^{\alpha_i}\} \cup \{V_j^{\beta_i}\}$ be called $\tilde{V}_1^i, \dots, \tilde{V}_{s_i+t_i}^i$. For $j < n$ define $V_j^i = \tilde{V}_j^i$ and define $V_n^i = \cup_{k=n}^{s_i+t_i} \tilde{V}_k^i$. Now define $V_l := \cup_i V_l^i$. Now suppose V_l bounds disks into the grope complement. Then inductively for each i , either A_i or B_i can be surgered to produce a disk, since there exists a j such that $V_j^{\alpha_i} \subset V_l^i$ or $V_j^{\beta_i} \subset V_l^i$ and hence for all i , ∂A_i or ∂B_i bounds a disk. Hence a half basis of the bottom stage bound imbedded disks and so surgery produces a spanning disk.

Definition 2.2 *A set of handles has the trivialization property iff when caps of these handles are abstractly added to the grope along the v_i curves in this set of handles, the grope becomes contractible. Another way to say this is if the caps are added to the grope in a standard unknotted model in \mathbb{R}^3 , iterated surgery along the caps produces an unknotting disk.*

So now we have a handlebody surrounding G , with n groups of handles satisfying the trivialization property. Such a group V_i is said to be *framed unlinked* if the \bar{v}_i bound disks whose interiors intersect the grope only at handles not associated to a core in V_i . This set of disks is called a *cap*. (When a disk does intersect a handle, by general position we can assume it does so in a single level $D^2 \times \{t\}$.) If V_i is not framed unlinked, we say it is *framed linked*. The reason for this terminology is that even if a group of handles $\{H_i\}$ look like an unlink, a pushed out core \bar{v}_i may link with v_i and hence will not be able to bound a disk into the grope complement.

Fix a projection of the grope so that the 1-manifolds with boundary, $\bar{V}_i \cap B$ are standardly arranged in decreasing order as the height function increases as in figure 4.

To show this is possible, let $F : (III) \times I \rightarrow S^2$ be an isotopy of $\cup(\bar{V}_i \cap B)$ to the standard picture depicted in figure 4. Put a collar $C_1 \cong S^2 \times I$ on B corresponding

to the isotopy F . Let $C_2 \cong S^2 \times I$ be a collar on $B \cup C_1$ corresponding to a constant isotopy. Let $C_3 \cong S^2 \times I$ be a collar on $B \cup C_1 \cup C_2$ corresponding to the isotopy inverse to F . We can think of the collar $C_1 \cup C_2 \cup C_3$ as an ambient isotopy of $\cup \overline{V}_i$ rel B . We can now take the new B to be $B \cup C_1$, and the new handles to be regular neighborhoods of the part of the isotoped grope outside the new ball. The new disks D_i are formed by evolving the \overline{V}_i outward along the radial parameter.

Definition 2.3 *A grope with the handlebody structure of theorem 2.1 and designated groups of handles satisfying the trivialization property, V_i , having a projection as described above, is said to be in standard position.*

2.2 The decorated graph $\Gamma(G)$ associated to an imbedded grope G .

Definition 2.4 *Given a grope G in standard position, we form a decorated graph $\Gamma(G)$ as follows. We call the vertices V_1, \dots, V_n , corresponding to the n groups of handles satisfying the trivialization property. We put an l next to a vertex if that group of handles is framed linked. We put an edge between V_i and V_j , $i < j$, if the group of handles V_j ever cross over the group V_i , with respect to the given projection.*

As an exercise, note that if $\Gamma(G)$ consists of vertices with no edges and no l 's, then ∂G is unknotted. This is because the cores all bound disks, and also they are stacked with V_1 above V_2 above V_3 , etc. Thus, in particular, there is a plane separating V_1 and V_2 intersecting the ball in a level circle with respect to the height function of the projection. So the disks bounding V_1 say can be restricted to lie above V_2 since if the disks ever ventured below the plane separating V_1 and V_2 they could be surgered to lie above that plane, using the 3-manifold topologist's favorite tool, the inner-most disk argument.

Definition 2.5 *Given a decorated graph Γ , the complexity, $c(\Gamma)$ is defined to be the number of edges, E , plus ξ , which is defined as the number of vertices decorated with an l . That is $c(\Gamma) = E + \xi$.*

Definition 2.6 *A group of vertices V_{i_1}, \dots, V_{i_k} is said to be free if the V_i are all framed unlinked and if for all $1 \leq s, t \leq k$ ($s \neq t$) there is no edge in the graph connecting V_{i_s} with V_{i_t} .*

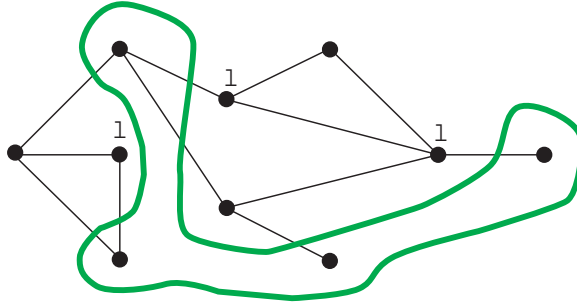


Figure 5: The circled vertices are a group of 4 free vertices.

3 Two types of moves and the reduction to the case where the bottom stage of G is genus 1

3.1 Move type I, a complexity reducing move

Given an edge or an ‘1’ in the graph Γ , we define a move which has the effect of deleting the edge or ‘1’, i.e. reducing $c(\Gamma)$. Suppose the edge is between V_i and V_j . That means that some of the handles in V_j cross over or under some of the handles in V_i in the wrong way. Then the move is defined to be the homotopy which switches these handle crossings, supported in balls associated to the crossings. (See [G] for instance.) In order to remove an ‘1’ from a vertex, suppose that vertex is V_i . To unknot a handle in V_i , first do handle crossings of the handle with itself so that the handle bounds a disk which intersects only other handles. However we must also make sure the handle is untwisted, which is to say that the pushed out core \bar{v}_i of the handle bounds a disk which intersects only other handles. Let the boundary of the disk that the handle bounds be the longitude. Then Dehn twist to remove the appropriate number of multiples of the meridian of the handle. This twist is supported in some small section of the handle $D^2 \times [a, b]$. Do this for every handle in V_i to remove the ‘1’.

Notice that any number of type I moves may be performed simultaneously, since the supports are by construction disjoint, with the effect that the corresponding edges or ‘1’s are deleted in Γ .

3.2 Move type II, moves on free vertices

Given a set, F , of k free vertices we define k moves as follows. Since the vertices are free, there are planes in S^3 which separate the groups of handles in F , and which intersect the ball of the grope’s handlebody in circles which lie standardly as level circles between the attaching regions of the groups of handles. We can now choose homotopies supported between the appropriate planes which contract the sets of handles down to trivial handles within a small neighborhood of the ball as in figure 6. These moves obviously have disjoint support by construction, and further doing any collection of them has the effect

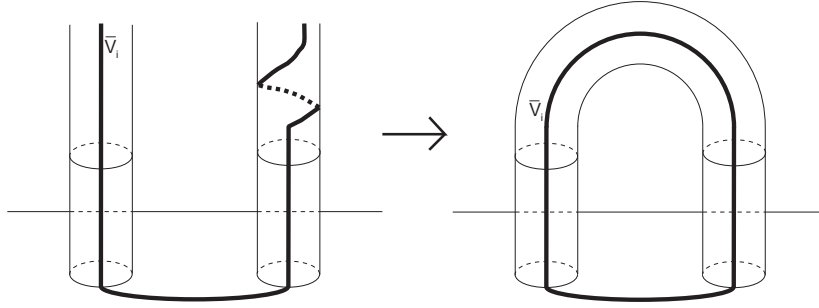


Figure 6: A type II move trivializing a handle.

of trivializing at least one set of handles with the trivialization property. This has the effect of trivializing the grope.

3.3 The graph $\tilde{\Gamma}$.

Given a grope G , we define a slightly different version of the graph defined in section 2.2. Fix a projection of the handlebody where all the $\bar{v}_i \cap B$ occur in increasing order as height decreases. For the graph, $\tilde{\Gamma}$, we let there be vertices v_i for every handle in the grope's associated handlebody, as opposed to one for each of the n groups of handles. We put an l next to the vertex if that core is framed linked in the previously defined sense (since it is just one core you might say *framed knotted* instead), and we draw an edge between two vertices if the corresponding handles cross in the wrong order in the projection.

In terms of the graph $\tilde{\Gamma}$ we can still do type \tilde{I} and type \tilde{II} moves, defined in the obvious analogous way. However, the result of doing a type \tilde{II} move is no longer necessarily to trivialize the knot but instead to reduce the total genus of the grope, where total genus is defined as the sum of the genera of all the stages of the grope. (Since a trivialized handle has a core which bounds a disk, one can iteratively surger along the successively produced disks as long as the successive stages are genus one. When you hit a higher genus stage, the surgery has the effect of lowering the genus of that stage by one.) We have thus proved the following lemma.

Lemma 3.1 *If $\tilde{\Gamma}(G)$ has k free vertices then $[\partial G]_{k-1} = \sum \pm [\partial G_i]_{k-1} \in \mathcal{G}_{k-1}$, where G_i is a grope of lower total genus than G , but of the same class.*

[Proof]

Let S be the scheme of type \tilde{II} moves defined above. Then $\sum_{\sigma \in S} (-1)^{|\sigma|} [\partial G_\sigma]_{k-1}$. If $\sigma \neq \emptyset$, then G modified by σ is of lower total genus as analyzed above. \square .

3.4 Genus 1 is sufficient

Consider, toward a contradiction, a counterexample which has minimal (total genus, $c(\tilde{\Gamma})$), ordered lexicographically. This example has bottom stage genus > 1 , by assump-

tion. Notice that $\tilde{\Gamma}$ has at least $2n$ vertices, since for each pair of dual basis elements in the bottom stage we get at least n vertices. I claim we can assume $c(\tilde{\Gamma}) \leq \lceil \frac{n}{2} \rceil$. Otherwise, consider a scheme, S , consisting of $\lceil \frac{n}{2} \rceil + 1$ type \tilde{I} moves.

By the triviality of the $Tot(K; S)$'s mentioned in the introduction, inside the group $\mathcal{G}_{\lceil \frac{n}{2} \rceil}$ the knot $K = \partial G$ is equivalent to a sum of knots of lower complexity and equal total genus,

$$[K]_{\lceil \frac{n}{2} \rceil} = - \sum_{\emptyset \neq \sigma \subset S} (-1)^{|\sigma|} [K_\sigma]_{\lceil \frac{n}{2} \rceil} \quad (2)$$

Each of these knots in the sum have reduced complexity, hence, by minimality is $\lceil \frac{n}{2} \rceil$ -trivial. Thus $[K]_{\lceil \frac{n}{2} \rceil} = 0$, contradicting that K is a counterexample.

So it suffices to consider knots with $c(\tilde{\Gamma}) \leq \lceil \frac{n}{2} \rceil$. Now

$$\begin{aligned} \xi + E &\leq \lceil \frac{n}{2} \rceil \\ \xi + 2n - \lceil \frac{n}{2} \rceil &\leq 2n - E \\ \Rightarrow \xi + \lceil \frac{n}{2} \rceil + 1 &\leq \chi(\tilde{\Gamma}), \end{aligned}$$

since $2n - \lceil \frac{n}{2} \rceil \geq \lceil \frac{n}{2} \rceil + 1$. On the other hand $\chi(\tilde{\Gamma}) = b_0 - b_1 = \# \text{ components} - \# \text{ cycles}$, implying there are at least χ components. Hence there exist at least $\xi + \lceil \frac{n}{2} \rceil + 1$ components, implying there are at least $\lceil \frac{n}{2} \rceil + 1$ components without any framed linked vertices. We can choose a free set of $\lceil \frac{n}{2} \rceil + 1$ vertices by selecting one vertex from each of these. So by lemma 3.1, $[\partial G]_{\lceil \frac{n}{2} \rceil} = \sum \pm [\partial G_i]_{\lceil \frac{n}{2} \rceil} = 0$ by the minimality of total genus. But this is a contradiction, since K was supposed to be a counterexample. We have thus proved the following lemma.

Lemma 3.2 *If all class n gropes with genus one bottom stage are $\lceil \frac{n}{2} \rceil$ -trivial, then all class n gropes are $\lceil \frac{n}{2} \rceil$ -trivial.*

From now on assume the bottom stage of the grope is genus one.

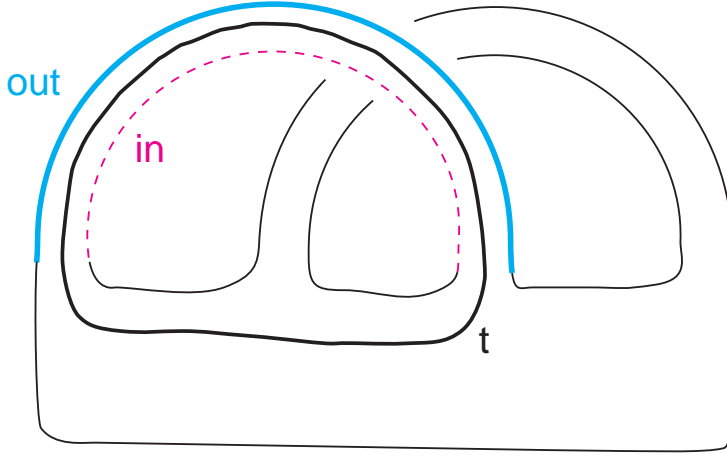


Figure 7: The “in” and “out” arcs.

4 Description of the In/Out Trick

4.1 Introduction

Whereas the two type of moves defined in the previous section preserve the grope structure, the move described in this section, the in/out trick, does not. However, the move is necessary to prove the optimal result about grope triviality. Indeed we also use it to construct the examples of section 6, for which lemma 4.1, section 4.3, is needed.

4.2 The in/out trick

Note that G divides naturally into two halves, the half attached to a particular band of the bottom stage, and that attached to the dual band. Assume from now on that the $\overline{V}_i \cap B$ are in standard position such that all the $\overline{V}_i \cap B$ on one half of the grope lie below all the $\overline{V}_i \cap B$ on the other half. In the handlebody, consider a framed unlinked $V_i = X$. Let $\Delta_{x_1}, \dots, \Delta_{x_m}$ be a cap. That is $\partial(\cup \Delta_{x_i}) = \overline{V}_i$, with the disks possibly intersecting the other handles. If the cap does not intersect the other handles, then ∂G is unknotted.

We consider two subarcs of $K = \partial G$ called “in” and “out” by coloring the bottom stage of G as in figure 7, where t is the curve to which X ’s half of the grope is attached. (In the inductive definition of the V_i , it is obvious that V_i lives in one half or the other.)

Suppose H is a handle intersecting $\Delta_{x_1} \cup \dots \cup \Delta_{x_m}$. Choose m arcs inside the Δ_{x_i} from $H \cap (\Delta_{x_1} \cup \dots \cup \Delta_{x_m})$ to X . The endpoint of each of these arcs lies on a handle of V_i at some slice $D^2 \times \{t_0\}$. The grope slice at this point looks like some Ξ_i . Push H along these arcs as in figure 8.

This introduces intersections of H with a top stage of the grope. (Although it’s just an isotopy of K .) Continue pushing through successive stages of the grope to eliminate the intersections, being sure to push them down to the next stage in a small neighborhood

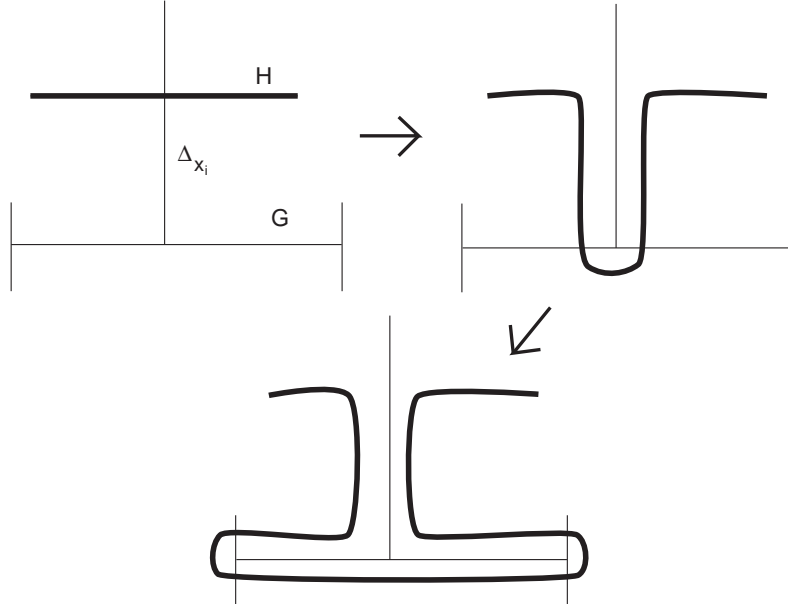


Figure 8: Successive pushes of H . In the pictured case, the X handle is locally modelled on $\Xi_2 \times I$.

of $D^2 \times \{t_0\}$. Continue doing this, for all handles, H_i , intersecting the Δ_{x_i} until you've pushed to the bottom stage, but don't push across the knot off the bottom stage (yet). If we push again, we'd be introducing actual crossing changes of the knot. This preliminary isotopy will be called *phase I* of the in/out trick. We define the “in” and “out” moves as in figure 9. These two homotopies are *phase II*. They are clearly disjointly supported after phase I.

Doing both $\text{in}X$ and $\text{out}X$ trivializes K , since it gives a grope with $X = V_i$ bounding disks. If we just do $\text{in}X$, then we can turn the grope which t bounds, G' , into a disk, Δ in a regular neighborhood of $G' \cup \cup_i \Delta_{x_i}$, by surgery. That is, glue in two parallel copies of the Δ_{x_i} to make that stage of G' a collection of disks. Iterate the procedure with these new disks until t bounds an (imbedded) disk, Δ . (After all, we just removed all intersections.) One subtlety is that this disk Δ will run through the handles H_i , but this doesn't matter. Now, since we've removed all intersections of H between the “in” arc and t , we can isotop the “in” arc along Δ to the arc μ as in figure 10. This is *phase III*. The “out” arc was never made to cross itself, so after the “in” arc trivializes to μ , the “out” arc can be isotoped back to its original position. But now the band dual to t pulls away, and we are left with G' . This final isotopy is *phase IV*. A similar analysis holds for doing $\text{out}X$, but one must pay attention to orientations. If t is oriented the same way as the “in” arc, then it will be oriented oppositely to the “out” arc. Hence after doing $\text{out}X$ we get the knot $\rho(\partial G')$, where ρ is the map reversing orientation. (Note: it is not known whether finite type invariants can ever distinguish a knot K from $\rho(K)$.)

For a genus one surface, the “in” and “out” arcs are symmetric so the move $\text{out}X$ gives the same (unoriented) result as $\text{in}X$. However, for a higher genus surface, the “out”

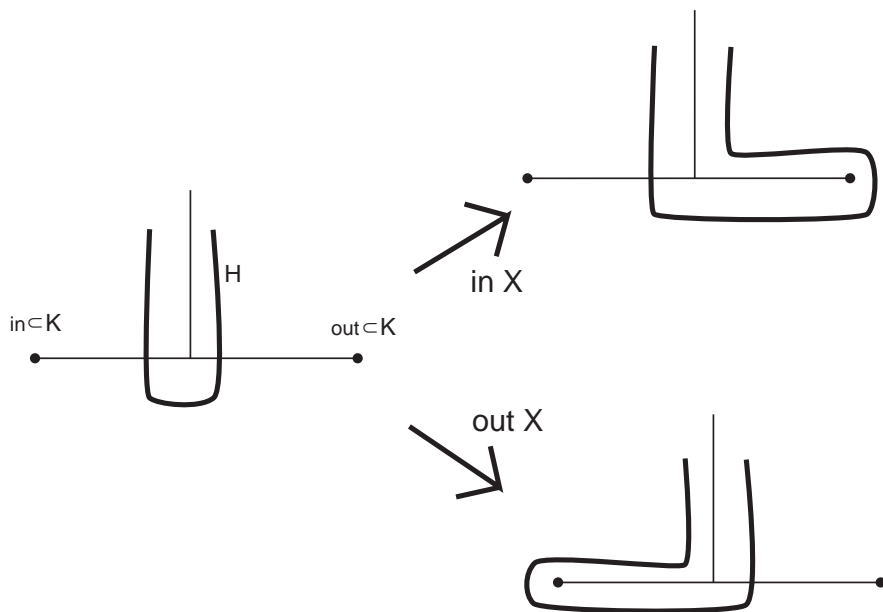


Figure 9: The “in” and “out” moves.

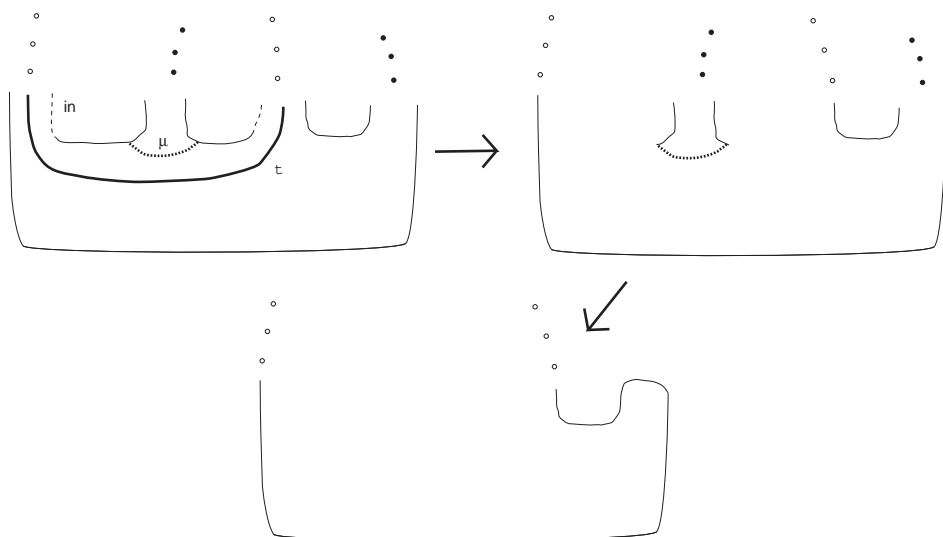


Figure 10: Doing $\text{in}X$ gives the knot t bounding the grope G' .

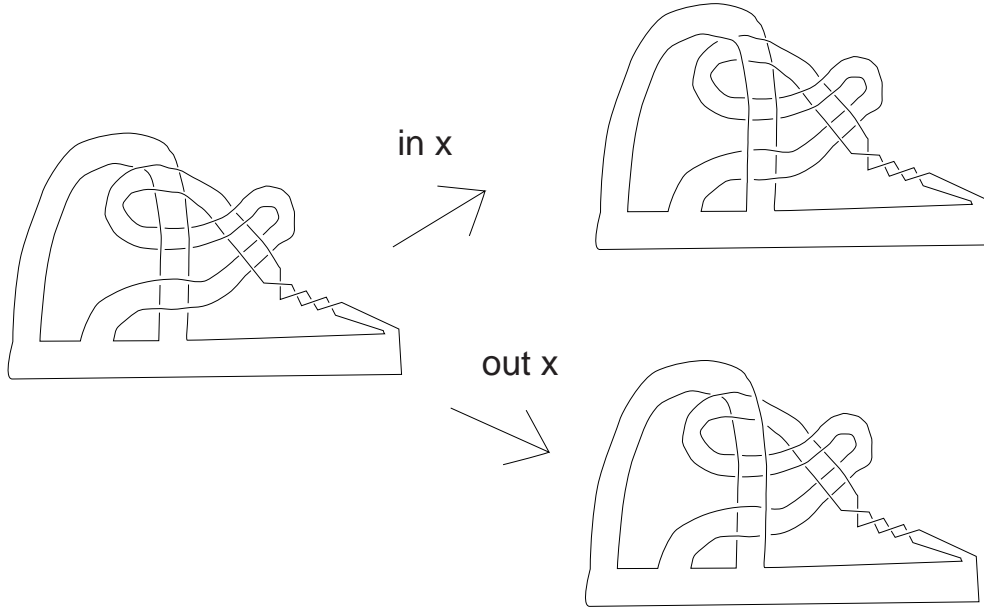


Figure 11: The knots on the right are unknots.

move no longer works, the problem occurring during phase IV, and this is why we need the bottom stage of G to be genus one.

4.3 Examples

We now use the in/out trick to give a proof that every knot is 1-trivial. This also follows from the main theorem and is well-known, but is good for illustrative purposes.

Suppose a knot, K , bounds a seifert surface with k pairs of dual bands $\{x_i, y_i\}_{i=1}^k$. Consider the scheme $S = \{s_1, s_2\}$ where s_1 is the move which unknots and untwists the x_1 band and also does crossing changes with other bands so that x_1 always crosses over them. s_2 does a similar thing for y_1 . Doing either s_1 or s_2 reduces the genus of the seifert surface and we are done inductively. Doing both gives a connected sum of a genus one knot that has unknotted bands, x, y , is 1-trivial. But the scheme $\{\text{inx}, \text{outx}\}$ now trivializes the knot. In this simple case, inx (respectively outx) may be visualized as the move making the “in” arc (respectively “out” arc) cross over everything in the projection. See figure 11.

We conclude this section with an interesting calculation which will be used in section 6.

Lemma 4.1 *Consider a grope G with genus one bottom stage which is formed by gluing the gropes G' and G'' to the bottom stage. They intersect in a point, $*$. There are two ways to resolve this intersection inside the bottom stage as pictured in figure 13. These*

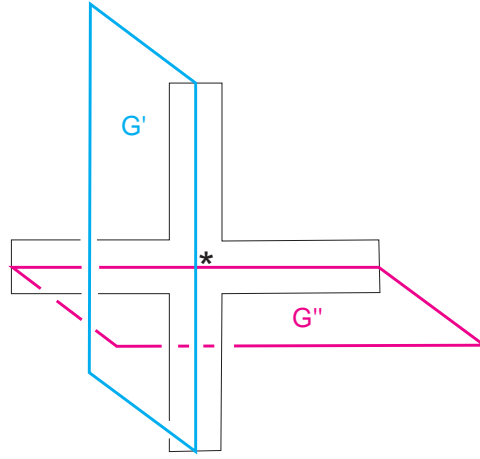


Figure 12: The local picture at the bottom stage of G .

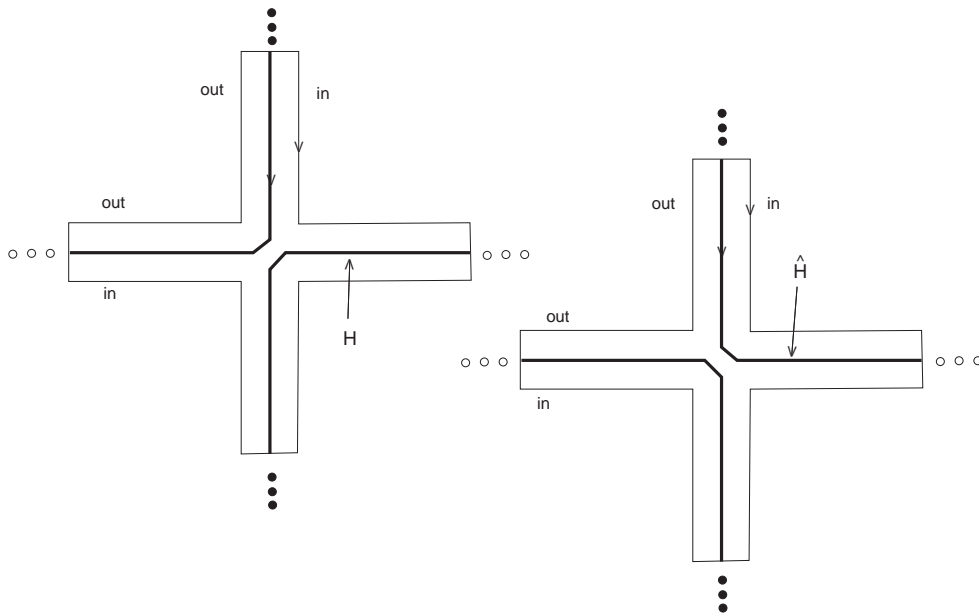


Figure 13: The two resolutions.

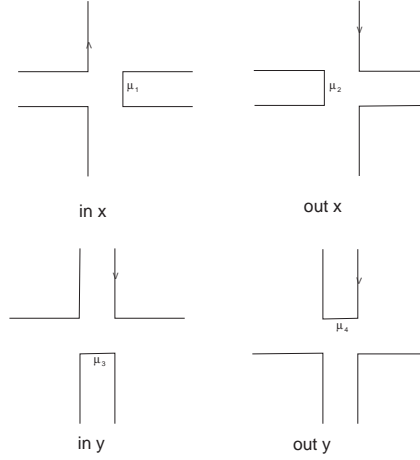


Figure 14: Moves in S .

give rise to 2 knots which are denoted H and \hat{H} . Let x be an unknotted vertex on the G half and y an framed unlinked vertex on the G'' half such that $\{x, y\}$ is not an edge. Consider the scheme $S = \{\text{inx}, \text{outx}, \text{iny}, \text{outy}\}$. Then $\text{Tot}(\partial G; S) = \sum_{\sigma \subset S} (-1)^{|\sigma|} \partial G_\sigma$, inside the monoid ring $\mathbb{Z}\text{Knots}$, is equal to $\partial G + H + \hat{H} + \rho(H) + \rho(\hat{H})$.

[Proof]

Consider figure 12 depicting a neighborhood of $G' \cap G''$. Note the various moves in S can be pictured as in diagram 14, the μ_i being the same as the μ arc previously considered. I claim the following:

$$\sum_{\sigma \subset S} (-1)^{|\sigma|} \partial G_\sigma = (\partial G) - (\partial G' + \partial G'' + \rho(\partial G') + \rho(\partial G'')) + (H + \rho(H) + \hat{H} + \rho(\hat{H})) - (\partial G' + \partial G'' + \rho(\partial G') + \rho(\partial G''))$$

which follows from the following facts: doing any single move in S will give the four terms of the second summand as was analyzed in section 4.2. The third summand follows from diagram 15 and the fact that doing inx , outx or iny , outy together trivialize the grope as analyzed in section 4.2. Of course some justification is needed for diagram 15.

We must analyze what happens when we do, say, both inx and iny . Let $G'_{I,II}$ (respectively $G''_{I,II}$) be G' (respectively G'') modified by phases I and II of inx and iny . Phase III of inx (respectively iny) is supported in a regular neighborhood of $G'_{I,II} \cup \text{xcap}$ (respectively $G''_{I,II} \cup \text{ycap}$). Note $(G'_{I,II} \cup \text{xcap}) \cap (G''_{I,II} \cup \text{ycap})$ is the point $*$ in figure 12. Hence the phase III isotopies are independent except near the end when the ‘in’ arc gets near $*$ soon to become the μ arc. So do the isotopies until they are close to $*$ as in figure 16. But 16 is just a different picture of 15: inx , iny .

The fourth summand follows in the same way as the third, by considering triplets of moves in S and is left as an exercise to the reader. Finally doing all moves in S trivializes

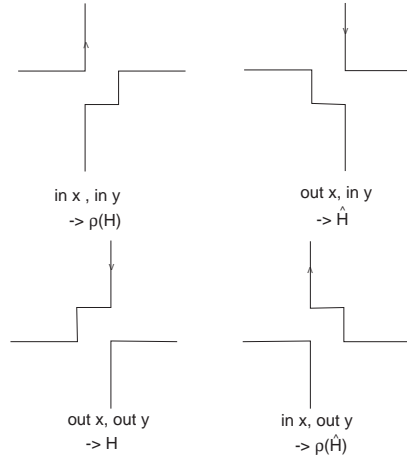


Figure 15: Several pairs of moves in S .

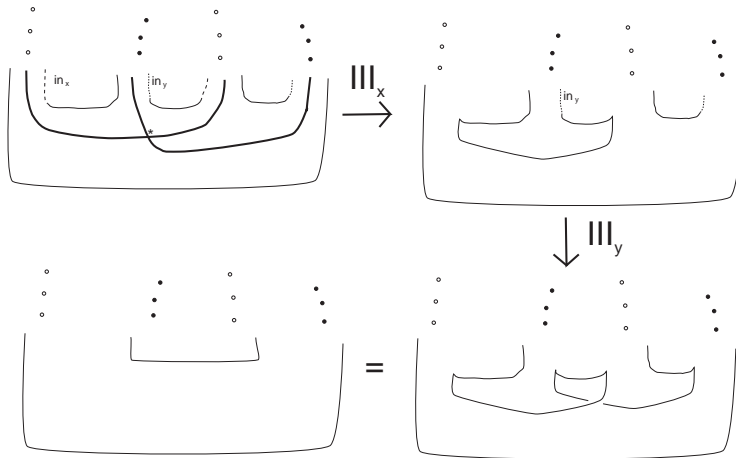


Figure 16: Doing an in move on each half of the grope.

the grope. \square

5 The Main Theorem

In this section, we prove the following

Theorem 5.1 *Every class n grope, G , is $\lceil \frac{n}{2} \rceil$ -trivial.*

[Proof]

We may assume $n = 2m + 1$ since the even case follows by thinking of a class $2m$ grope as a class $2m - 1$ grope by forgetting a stage. Also, we may assume $c(\Gamma) \leq m + 1$ since we have $m + 2$ moves in hand to reduce complexity. Now a set of m free vertices exists by the following euler characteristic argument. (b_i denote Betti numbers.)

$$\begin{aligned} c(\Gamma) &\leq m + 1 \\ \xi + E &\leq m + 1 \\ \xi + m &\leq 2m + 1 - E = \chi(\Gamma) \\ \xi + m &\leq b_0 - b_1 \\ b_0 - \xi &\geq m + b_1 \end{aligned}$$

Hence there are at least m connected components of Γ which have no framed linked vertices. Picking a vertex from each such component yields the desired m free vertices.

In order to proceed, we need the following interesting lemma. Let \mathcal{V} be the set of vertices of our grope.

Lemma 5.1 *Suppose $F \subset \mathcal{V}$ is a set of m free vertices, $F = \{v_1, \dots, v_m\}$. We can assume $c(\Gamma \setminus \text{star} F) = 0$. That is, if we remove F and all edges which hit F from Γ , the complexity of the resulting graph is 0.*

[Proof]

Suppose otherwise. Let G be a class $2m + 1$ grope with a set of m free vertices, F , contradicting the claim, with $c(\Gamma \setminus \text{star} F)$ minimal. By hypothesis this number is bigger than zero. Let $S = \{s_1, \dots, s_{m+2}\}$ be the scheme in which s_1, \dots, s_{m-1} are type II moves trivializing the v_1, \dots, v_{m-1} handles supported between separating planes. s_m, s_{m+1} are the in and out move respectively on the v_m handles. These two moves are supported in a neighborhood of the v_m handles with caps, which is separated from the v_1, \dots, v_{m-1} handles by hyperplanes, and so is disjointly supported from the type II moves. Finally, s_{m+2} is a type I move which reduces $c(\Gamma \setminus \text{star} F)$. It is possible that $\text{supp}(s_{m+2}) \cong \mathbb{I}D^3$ is not disjoint from the other moves, since the separating planes may intersect this disjoint union of balls. However, since s_{m+2} is only reducing complexity away from v_1, \dots, v_m , at least the handles v_1, \dots, v_m do not hit $\text{supp}(s_{m+2})$. But then the separating planes are easily pushed out of $\text{supp}(s_{m+2})$ using the balls to guide the isotopy, say. It is then an easy matter to separate these balls from the other moves.

So $\sum_{\sigma \in S} (-1)^{|\sigma|} [\partial G]_{m+1} = 0$, and let us see what this says. In preparation, let us suppose that G is formed by attaching the gropes H' and H'' to the dual bands of the bottom stage, thereby partitioning \mathcal{V} into two nonempty sets $\mathcal{V}_{H'}$ and $\mathcal{V}_{H''}$. Suppose without loss that $v_m \in \mathcal{V}_{H'}$. Let $S_{H'}$ and $S_{H''}$ partition $\{s_1, \dots, s_{m-1}\}$ into two sets in the obvious way. Let $S_I = \{s_m, s_{m+1}\}$ and $S_C = \{s_{m+2}\}$.

Note that we can assume s_{m+2} reduces $c(\Gamma \setminus \mathcal{V}_{H'})$ since if this complexity were zero, then $\mathcal{V}_{H''}$ would have no edges hitting it, (and no framed linked vertices). By the earlier stated assumption that the height function separates the two halves of the grope H' and H'' , the handles on the H'' half all bound disks, implying of course that the grope is trivial contradicting that G is a counterexample. Thus we can assume some complexity not contained wholly within the H' half, and without loss s_{m+2} reduces this.

We are now in a position to describe what happens under the various combinations of moves from $S_{H'}$, $S_{H''}$, S_I and S_C , with the initial assumption that neither $S_{H'}$ nor $S_{H''}$ is empty. In the following list of cases, case i refers to a set of moves, σ , which hits i of the above 4 sets.

Case 0

This is the empty move yielding ∂G .

Case 1

By our previous analysis of the handle trivializing moves, if $\sigma \subset S_{H'}$ or $\sigma \subset S_{H''}$, ∂G_σ is the **unknot**. $K_{s_{m+2}}$ has less of the appropriate complexity so by minimality $[K_{s_{m+2}}]_{m+1} = \mathbf{0}$. The left over terms are the ones gotten from the in/out trick: doing both of s_m, s_{m+1} is the **unknot**, while $K_{s_m}, K_{s_{m+1}}$ are $\partial H'$ and $\rho(\partial H')$.

Case 2

σ hits $S_{H'}, S_{H''}$: **unknot**.

σ hits $S_{H'}, S_I$: $S_{H'}$ trivializes some handles, and then s_m or s_{m+1} give H' with trivialized handles, an **unknot**. Doing both the in and out move also yields an **unknot**.

σ hits $S_{H'}, S_C$: $S_{H'}$ trivializes handles of the grope $G_{s_{m+2}}$ yielding an **unknot**.

σ hits $S_{H''}, S_I$: $S_{H''}$ gives some grope with the H' half unaltered. Doing one move from S_I then gives the H' half. Specifically, $\sum_{\emptyset \neq \tau \subset S_{H''}} (-1)^{|\tau|} \{[\partial H']_{m+1} + [\rho(\partial H')]\}_{m+1}$. Again if we do both s_m and s_{m+1} the result is obviously an **unknot**.

σ hits $S_{H''}, S_C$: **unknot**.

σ hits S_I, S_C : S_C gives some grope with the H' half unaffected. So as in a previous case we get $\partial H' + \rho(\partial H'')$.

Case 3

σ hits $S_{H''}, S_I, S_C$: $S_{H''}, S_C$ give a grope with H' half intact, and so as in two of the previous cases we get, adjusting the sign to include the s_{m+2} move,

$$\sum_{\emptyset \neq \tau \subset S_{H''}} (-1)^{|\tau|+1} \{[\partial H']_{m+1} + [\rho(\partial H')]\}_{m+1}.$$

σ hits $S_{H'}, S_I, S_C$: **unknot**.

σ hits $S_{H'}, S_{H''}, S_C$: **unknot**.

σ hits $S_{H'}, S_{H''}, S_I$: **unknot**.

Case 4

This involves doing at least one move from each group and is an **unknot**.

We conclude

$$\begin{aligned} \sum_{\sigma \in S} (-1)^{|\sigma|} [\partial G_\sigma]_{m+1} &= [\partial G] - [\partial H'] - [\rho(\partial H')] + \sum_{\emptyset \neq \tau \subset S_{H''}} (-1)^{|\tau|} \{[\partial H'] + [\rho(\partial H')]\} \\ &\quad + [\partial H'] + [\rho(\partial H')] + \sum_{\emptyset \neq \tau \subset S_{H''}} (-1)^{|\tau+1|} \{[\partial H'] + [\rho(\partial H')]\} = [\partial G]_{m+1} = 0 \end{aligned}$$

This is a contradiction.

If $S_{H'} = \emptyset$, then only cases leading to an $m+1$ -trivial knot are eliminated so the calculation still goes through.

If $S_{H''} = \emptyset$, then two nontrivial cases are eliminated: the $S_{H''}, S_I$ subcase of case 2 and the $S_{H''}, S_I, S_C$ subcase of case 3. The calculation is now $\sum_{\sigma \in S} (-1)^{|\sigma|} [\partial G_\sigma]_{m+1} = [\partial G] - [\partial H'] - [\rho(\partial H')] + [\partial H'] + [\rho(\partial H')] = 0$ which still achieves the desired result $[\partial G]_{m+1} = 0$. \square

Continuing the proof of theorem (5.1), recall we had found a free set of m vertices F . But the preceding lemma proves that $\mathcal{V} \setminus F$ can also be assumed free, this time of cardinality $m+1$. Indeed we may assume that for any free F' of cardinality m , $\mathcal{V} \setminus F'$ is also free. This actually implies $c(\Gamma) = 0$ and therefore that G is trivial, and we are done: since $F, \mathcal{V} \setminus F$ are free, all framed linked vertices have been eliminated. Suppose $\mathcal{V} \setminus F = \{w_1, \dots, w_{m+1}\}$. Let $F' = \{w_1, \dots, w_m\}$. Then $F \cup \{w_{m+1}\}$ must be free, implying w_m shares an edge with no vertex in F . Since it shared none with $\mathcal{V} \setminus F$, w_{m+1} is in fact isolated. But then by symmetry all of $\mathcal{V} \setminus F$ is isolated. Since the only edges were between $\mathcal{V} \setminus F$ and F , there are no edges whatsoever. \square

6 Showing the Bound is Sharp

In the following section we show that for all $n \geq 2$ there are knots bounding gropes of class n which are not $\lceil \frac{n}{2} \rceil + 1$ -trivial. In fact, we find K such that $J_K^{(\lceil \frac{n}{2} \rceil + 1)}(1) \neq 0$, where $J_K(t)$ is the Jones polynomial. It is well known that the j th derivatives of the Jones polynomial evaluated at 1 are type j invariants. Note that $J_{\bullet}^{(m)}(1)$ is not additive under connect sum (primitive), but is easily seen to be additive on $m - 1$ -trivial knots.

For this section, it is convenient to use a different graph than the one we used previously.

Definition 6.1 *Let G be a grope of class n in standard position with framed unlinked handles bounding fixed caps. We define the graph $\Gamma_{\Delta}(G)$ as follows. The vertices as before correspond to the V_i , the n collections of handles satisfying the trivialization property. We put in an edge between V_i and V_j if the corresponding caps intersect.*

Note that type II moves on a free set of vertices have their obvious analog in this setting: we make the moves by using the cap to guide the homotopy. We call these *type II $_{\Delta}$* moves, for clarity. The moves are then obviously disjointly supported since the caps are hypothesized to be disjoint.

We prove the following statement inductively:

Theorem 6.1 *For all even n , there is a grope G of class n with all the cores V_i unknotted, such that the corresponding graph Γ_{Δ} has no first homology, and such that each vertex has valence less than or equal to 2. Further the edges ending in valence 1 vertices correspond to finger moves in the following sense: Let $\{V_i, V_j\}$ be an edge with V_i valence 1. Suppose $V_i = \{v_i^1, \dots, v_i^{l_i}\}$, $V_j = \{v_j^1, \dots, v_j^{l_j}\}$. Then we insist the \bar{v}_i^k (resp. \bar{v}_j^k) bound disks Δ_i^k (resp. Δ_j^k) such that each Δ_i^k intersects exactly one Δ_j^k in a single clasp singularity. Each Δ_j^k disk is hit at most once by the Δ_i^k disks.*

This grope satisfies $J_K^{(\lceil \frac{n}{2} \rceil + 1)}(1) \neq 0$.

This is sufficient for our purposes since it also implies the odd case. Just think of a grope of class $2m$ as a grope of class $2m - 1$ by ignoring one of the top stages. Since $\lceil \frac{2m}{2} \rceil = \lceil \frac{2m-1}{2} \rceil = m$, any example of class $n = 2m$ with $J^{(\lceil \frac{n}{2} \rceil + 1)}(1) \neq 0$ is also an example as a class $n = 2m - 1$ grope.

[Proof]

If $n > 2$, note that a graph satisfying the induction hypothesis will have $\lceil \frac{n}{2} \rceil$ free vertices and two special disjoint edges each containing a vertex which is not contained in any other edge. To see this note that such a graph is contained in a graph which is homeomorphic to an interval, the free vertices being alternating vertices in this graph, and the special edges being the edges at the ends of the interval.

We need the base cases $n = 2$, see figure (17), and $n = 4$, the second of which we defer to the end since we build the $n = 4$ example from the $n = 2$ example using a construction of the proof. The $n = 2$ example does not suffice because in order to get the induction going we need the graph to have at least 2 edges.

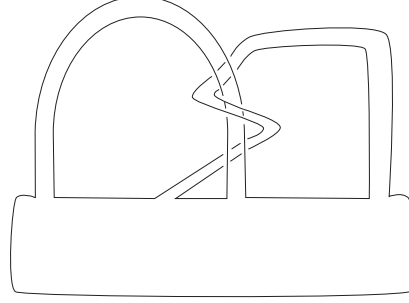


Figure 17: This knot has $J(t) = 2 - t + t^2 - 2t^3 + t^4 - t^5 + t^6$, as calculated by Knotscape, with $J^{(2)}(1) = 12 \neq 0$.

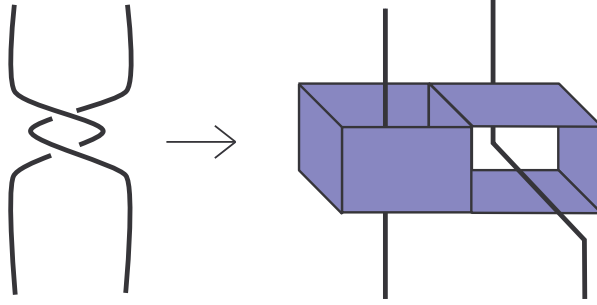


Figure 18: Unlinking two handles and relinking them with the torus T_α .

Now assume G is such a grope satisfying the statement of theorem 6.1 for $n = 2m$. Suppose the two special edges have endpoints V_i, V_j and V_l, V_m respectively, with V_i and V_l the “dangling” vertices. Take the edge $\{V_i, V_j\}$ in $\Gamma(G)$ and delete it, that is unlink the corresponding pairs of handles of G . Link each pair of these handles with a punctured torus T_α as in figure 18.

Notice that when the pairs of handles are pushed across each other to relink, the boundary of the punctured tori will bound a symmetric surgery disk. Denote by \tilde{G} the grope G modified as in figure 18, that is with the edge $\{V_i, V_j\}$ deleted. Connect the boundaries of the punctured tori, T_α , with the bottom stage of \tilde{G} by some bands disjoint from the rest of the T_α and from \tilde{G} and also disjoint from the various caps associated to all the vertices of \tilde{G} . Call this new grope H_{ij} . If $J_{H_{ij}}^{(m+1)} \neq 0$ let $H = H_{ij}$ and proceed.

Otherwise, carry out the same procedure for the edge $\{V_l, V_m\}$. If this also fails, i.e. $J_{H_{lm}}^{(m+1)}(1) = 0$, we form the grope H_{lmij} , which is the grope gotten from doing the above procedure to *both* edges. Consider the scheme $\{s_1, \dots, s_{m-2}, x_{ij}, y_{lm}, z_{ij}, z_{lm}\}$, where the s_i are type II moves trivializing the i handles corresponding to vertices in the complement of the special edges, and where the x ’s, y ’s and z ’s are given on the corresponding T_α as pictured in figure (19). The added torus, T_α , has two bands x_α and y_α each linking a handle of \tilde{G} exactly once. The move x has the effect of removing the linkage of the

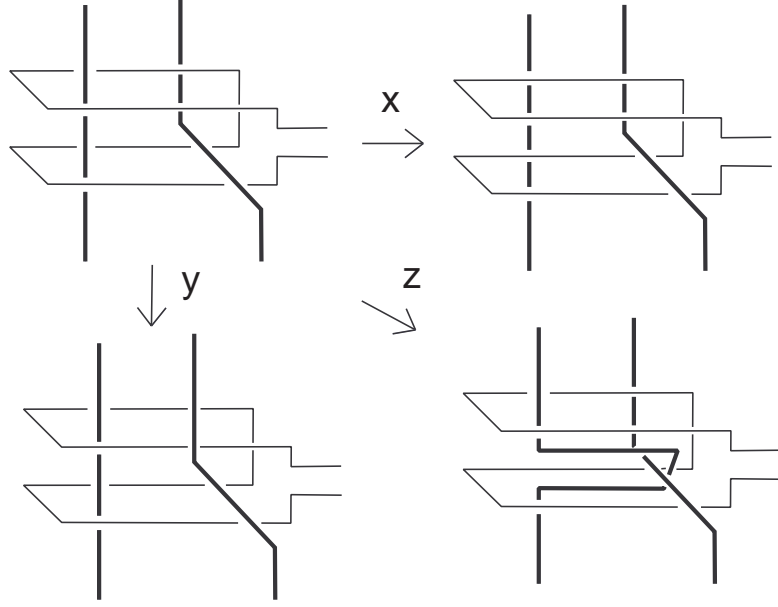


Figure 19: The x , y and z moves.

appropriate handle with x_α for all α , whereas y has the corresponding effect on the y_α . Indeed, as the reader may verify, doing any combination of these three moves x, y, z on a particular T_α causes this added torus to become compressible. There is a choice in which bands are called x_α and which y_α . Denote by x_α those bands which links V_i or V_m , and the y_a are then those bands linking V_j or V_l .

Now

$$-[H_{lmij}]_{m+1} = \sum_{\emptyset \neq \sigma \subset \{x_{ij}, y_{lm}, z_{ij}, z_{lm}\}} (-1)^{|\sigma|} [H_{lmij\sigma}]_{m+1} \quad (3)$$

$$= \sum_{\emptyset \neq \sigma \subset \{z_{ij}, z_{lm}\}} [H_{lmij}]_{m+1} \quad (4)$$

$$= -[H_{lm}]_{m+1} - [H_{ij}]_{m+1} + [G]_{m+1} \quad (5)$$

Here (3) follows since doing any of the s_i even in conjunction with other moves in the scheme will cause the \tilde{G} half to trivialize, followed by the t_α . (4) follows since doing either of x_{ij} or y_{lm} causes there to be a trivial group of handles corresponding to a vertex in the \tilde{G} half which then trivializes the grope. Finally, (5) follows since doing z_{ij} , say, relinks the ij handles while causing the appropriate T_α to compress, leaving the T_α linking with the l, m handles, i.e. H_{lm} .

From (5) we could immediately conclude (6), despite the fact that $J^{(m+1)}(1)$ is not in general additive in view of section 1.2, (1). However, since we need it later anyway, we will prove that G, H_{lmij}, H_{lm} , and H_{ij} are all m -trivial. Well G is m -trivial by the main theorem. H_{ij} is m -trivial: Let s_1, \dots, s_{m-1} be type II_Δ moves corresponding to

free vertices in the complement of $\{V_i, V_j\}$. (Their existence is proven later.) Consider the scheme $S = \{s_1, \dots, s_{m-1}, x, y\}$. Obviously, any subset of these trivializes H_{ij} . Symmetrically H_{ij} is m -trivial. But (5) indicates that H_{ijlm} is $m+1$, hence m , equivalent to a sum of m -trivial knots. It is therefore m -trivial itself.

Thus

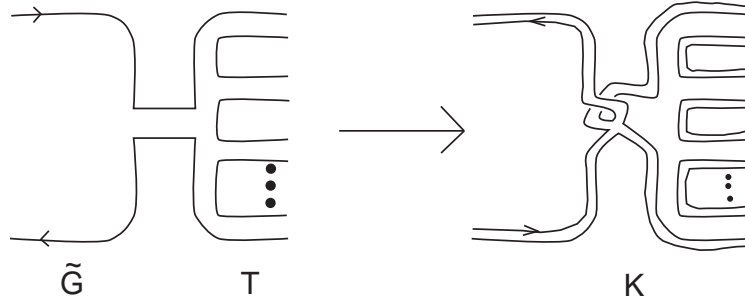
$$J_{H_{lmij}}^{(m+1)}(1) = J_{H_{lm}}^{(m+1)}(1) + J_{H_{ij}}^{(m+1)}(1) - J_G^{(m+1)}(1) \quad (6)$$

$$= 0 + 0 - J_G^{(m+1)}(1) \neq 0 \quad (7)$$

We may let $H = H_{lmij}$.

Recall that the T_α are connected via bands to \tilde{G} . We had a lot of choice in choosing these and may assume they are organized as follows. The T_α are band connect summed together to form T , which is then connected by a band with \tilde{G} which it links geometrically.

We form a class $n+2$ grope K from H by plumbing as follows:

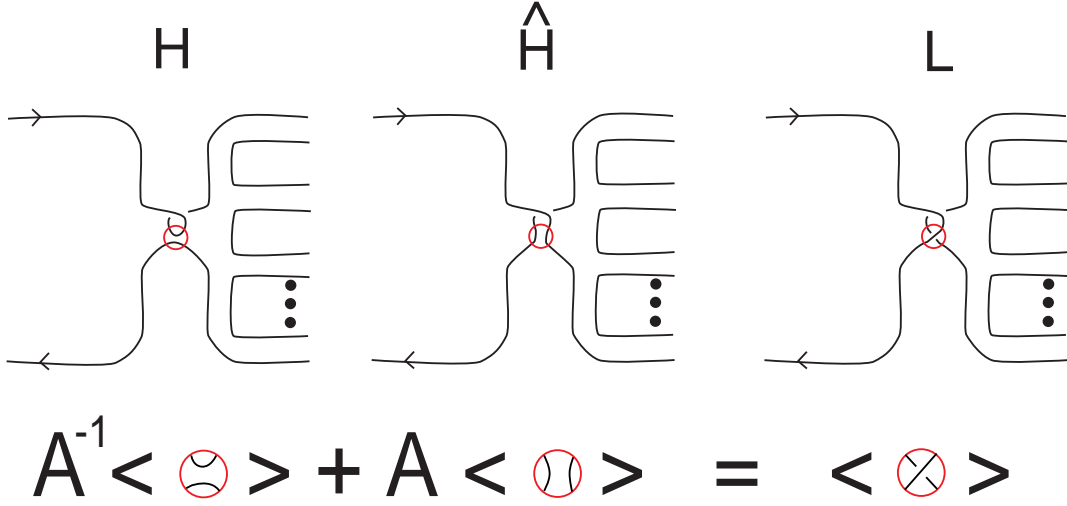


That is, K is formed by running a perpendicular annulus along \tilde{G} and one along T , and then plumbing these two annuli together to get a punctured torus, the bottom stage of a new grope. K is a class $n+2$ grope, the bottom stage of which has a core bounding a class n grope which was gotten from G , and the dual core of which bounds a class 2 grope which is the connected sum of the punctured tori, T_α . We claim $J_K^{(m+2)}(1) \neq 0$ which will complete the inductive step since $\lceil \frac{n+2}{2} \rceil + 1 = \lceil \frac{n}{2} + 1 \rceil + 1 = \lceil \frac{n}{2} \rceil + 2 = m+2$. Let x and y denote half symplectic bases of T . Then the altered graph is as in figure (21). Suppose $\{s_i\}_{i=1}^m$ are m free vertices on the G half of K , none of which is on an edge connected to x . Consider the scheme $S = \{s_i\}_{i=1}^{m-1} \cup \{\text{inx}, \text{outx}, \text{ins}_m, \text{outs}_m\}$, where the s_i are type II moves making the respective handles of K bound disks. As we know, if we do any of the s_i , then K trivializes. Thus,

$$-[K]_{m+2} = \sum_{\emptyset \neq \sigma \subset \{\text{inx}_m, \text{outx}_m, \text{inx}, \text{outx}\}} (-1)^{|\sigma|} [K_\sigma]_{m+2} \quad (8)$$

$$= [H] + [\hat{H}] + [\rho(H)] + [\rho(\hat{H})] \quad (9)$$

Where 9 follows from lemma 4.1. Let us compute the relationship between H and \hat{H} in terms of the Jones polynomial.



That is $A \langle \hat{H} \rangle + A^{-1} \langle H \rangle = \langle L \rangle$, where $\langle \bullet \rangle$ denotes the Kauffman bracket. Assume the writhe w of the diagrams is zero away from the pictured spots. Then $w(H) = 1, w(\hat{H}) = -1$ and $w(L) = 0$. Thus $J_H = (-A)^{-3} \langle H \rangle$, $J_L = \langle L \rangle$ and $J_{\hat{H}} = (-A)^3 \langle \hat{H} \rangle$. This implies the following relation, where we make the substitution $A^{-2} = t^{\frac{1}{2}}$,

$$-t^{\frac{1}{2}} J_{\hat{H}}(t) - t^{-\frac{1}{2}} J_H(t) = J_L(t) \quad (10)$$

Setting $u = t^{\frac{1}{2}}$, we get

$$u J_{\hat{H}}(u) - u^{-1} J_H(u) = J_L(u) \quad (11)$$

Note that \hat{H} is also m -trivial. We now analyze the triviality of L .

Claim 6.1 $J_L^{(k)}(1) = J_{Unlink}^{(k)}(1)$ for all $k \leq m + 2$.

[Proof]

First, choose m free vertices x_1, \dots, x_m on the \tilde{G} half of L , none of which shares an edge with the x vertex in $\Gamma_{\Delta}(K)$. The possible ways that we altered the graph are listed in figure 21. Each is contained in a graph which spans all vertices and is homeomorphic to a line as remarked earlier, which we can always choose with x as the second vertex from an endpoint. It is then obvious we can choose m vertices which do not share an edge with x .

Without loss, the local pictures of the tori T_{α} look like figure 20, with the V_i or V_m cap Δ_i or Δ_m hitting T_{α} as indicated.

Consider the two indicated sets of crossing changes, where ‘ x ’ is our old friend. Doing x' makes all V_i bound their caps in the complement of T . \tilde{G} is thus isotopic to the unlink in the complement of T since the isotopy is supported in a neighborhood of $\tilde{G} \cup V_i$ caps. We are then left with an unlink since T is unknotted when \tilde{G} is removed. Similarly doing x leaves an unlink of two components. Consider the scheme

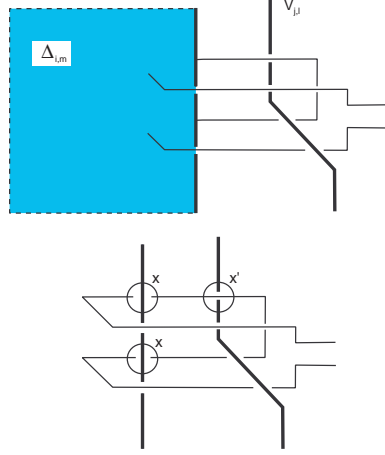


Figure 20: The x and x' moves.

$S = \{s_1, \dots, s_{m-1}, \text{in}x_m, \text{out}x_m, x, x'\}$, where s_i are type II_Δ moves trivializing the x_i . Doing such a type II_Δ move also yields an unlink, because after the x_i handles are trivialized, they bound caps in the complement of T and so the previous argument goes through. Similarly, the in and out moves on x_m trivialize the knot in a neighborhood of $\tilde{G} \cup x_m \text{cap}$. Indeed doing any combination of moves in S gives an unlink. Since $|S| = m + 3$, it follows that $J_L^{(k)}(1) = J_{\text{unlink}}^{(k)}(1)$ for all $k \leq m + 2$. (Recall $J^{(k)}$ is a type k link invariant. \square)

Applying $(\frac{d}{dt})^{m+2} = (\frac{du}{dt})^{m+2}(\frac{d}{du})^{m+2}$ to both sides of (11), and using claim 1, we get

$$-\left(\frac{d}{du}\right)^{m+2} (uJ_{\hat{H}}(u))(1) - \left(\frac{d}{du}\right)^{m+2} (u^{-1}J_H(u))(1) = \left(\frac{d}{du}\right)^{m+2} (J_{\text{unlink}}(u))(1) \quad (12)$$

To evaluate each of these, note $J_{\text{unlink}} = -A^{-2} - A^2 = -u - u^{-1}$. So the right hand side of (12) is equal to $-(\frac{d}{du})^{m+2}(u^{-1})|_1$. Also

$$\begin{aligned} \left(\frac{d}{du}\right)^{m+2} (uJ_{\hat{H}}(u))(1) &= \sum_{k=0}^{m+2} \binom{m+2}{k} \left(\frac{d}{du}\right)^k (u)|_1 \left(\frac{d}{du}\right)^{m+2-k} (J_{\hat{H}})_1 = \\ &J_{\hat{H}}^{(m+2)}(1) + (m+2)J_{\hat{H}}^{(m+1)}(1). \end{aligned}$$

Finally,

$$\begin{aligned}
& \left(\frac{d}{du} \right)^{m+2} (u^{-1} J_H) \\
&= \sum_{k=0}^{m+2} \binom{m+2}{k} \left(\frac{d}{du} \right)^k (u^{-1})|_1 \left(\frac{d}{du} \right)^{m+2-k} (J_H)(1) \\
&= J_H^{(m+2)}(1) + (m+2) \frac{d}{du} (u^{-1})(1) J_H^{(m+1)}(1) + \left(\frac{d}{du} \right)^{m+2} (u^{-1})|_1 J_H^{(0)}(1) \\
&= J_H^{(m+2)}(1) - (m+2) J_H^{(m+1)}(1) + \left(\frac{d}{du} \right)^{m+2} (u^{-1})|_1
\end{aligned}$$

Thus equation (12) becomes

$$- J_{\hat{H}}^{(m+2)}(1) - (m+2) J_{\hat{H}}^{(m+1)}(1) - J_H^{(m+2)}(1) + (m+2) J_H^{(m+1)}(1) = 0 \quad (13)$$

I claim that $J_{\hat{H}}^{(m+2)}(1) + J_H^{(m+2)}(1) \neq 0$. Otherwise, (13) implies that $J_{\hat{H}}^{(m+1)}(1) = J_H^{(m+1)}(1)$. Consider (9). It implies that $J_{K^{-1}}^{(k)}(1) = J_{H \# \rho(H) \# \hat{H} \# \rho(\hat{H})}^{(k)}(1)$ for all $k \leq m+2$. But K is $m+1$ -trivial by the main theorem, hence $J_{H \# \rho(H) \# \hat{H} \# \rho(\hat{H})}^{(m+1)}(1) = 0$. Since H and \hat{H} are m trivial, and J is invariant under ρ , this implies $2J_H^{(m+1)}(1) + 2J_{\hat{H}}^{(m+1)}(1) = 0$. This would imply $J_H^{(m+1)}(1) = J_{\hat{H}}^{(m+1)}(1) = 0$, contradicting our choice of H .

So $J_{\hat{H}}^{(m+2)}(1) + J_H^{(m+2)}(1) \neq 0$. Note $J_{H \# \rho(H) \# \hat{H} \# \rho(\hat{H})} = J_H^2 J_{\hat{H}}^2$. So

$$J_{H \# \dots \# \rho(\hat{H})}^{(m+2)}(t) = \quad (14)$$

$$\left(\frac{d}{dt} \right)^{m+2} (J_H^2(t) J_{\hat{H}}^2(t)) = \sum_{k=0}^{m+2} \binom{m+2}{k} \left(\sum_{l=0}^k \binom{k}{l} \left(\frac{d}{dt} \right)^l J_H \left(\frac{d}{dt} \right)^{k-l} J_H \right) \times \quad (15)$$

$$\left(\sum_{l=0}^{m+2-k} \binom{m+2-k}{l} \left(\frac{d}{dt} \right)^l J_{\hat{H}} \left(\frac{d}{dt} \right)^{m+2-k-l} J_{\hat{H}} \right). \quad (16)$$

In order for $\left(\frac{d}{dt} \right)^{k-l} J_H(1) \neq 0$, $k-l \in \{0, m+1, m+2\}$, which means that either $(k=l)$ or $(l=0 \text{ and } k=m+1)$ or $(k=m+2 \text{ and } (l=0 \text{ or } l=1))$. In order for $\left(\frac{d}{dt} \right)^l J_H(1) \neq 0$, $l \in \{0, m+1, m+2\}$. So the only potentially nonzero terms that arise upon evaluating (16) occur when $k=l=0, m+1, m+2$ or $k=m+1, l=0$ or $k=m+2, l=0$. So we get, for $k=0$,

$$\begin{aligned}
& (J_H(1)J_H(1)) \left(\sum_{l=0}^{m+2} \binom{m+2}{l} \left(\frac{d}{dt} \right)^l \Big|_1 J_{\hat{H}}(t) \left(\frac{d}{dt} \right)^{m+2-l} \Big|_1 J_{\hat{H}}(t) \right) = \\
& 1 \cdot 1(J_{\hat{H}}(1)J_{\hat{H}}^{(m+2)}(1) + J_{\hat{H}}^{(m+2)}(1)J_{\hat{H}}(1)) = \\
& 2J_{\hat{H}}^{(m+2)}(1).
\end{aligned}$$

For $k = m + 1$, we get

$$\begin{aligned}
& (m+2) \left(J_H(1)J_H^{(m+1)}(1) \right) \left(\sum_{l=0}^1 \binom{1}{l} \left(\frac{d}{dt} \right)^l \Big|_1 J_{\hat{H}}(t) \left(\frac{d}{dt} \right)^{1-l} \Big|_1 J_{\hat{H}}(t) \right) \\
& = (m+2)J_H^{(m+1)}(1) \cdot 0 = 0
\end{aligned}$$

For $k = m + 2$, we get

$$\begin{aligned}
& \left(J_H(1)J_H^{(m+2)}(1) + J_H^{(m+2)}(1)J_H(1) \right) \cdot (J_{\hat{H}}^{(0)}(1)J_{\hat{H}}^{(0)}(1)) = \\
& 2J_H^{(m+2)}(1).
\end{aligned}$$

So, equation (9) implies $2J_H^{(m+2)}(1) + 2J_{\hat{H}}^{(m+2)}(1) = J_{K^{-1}}^{(m+2)}$. Thus, since $J_H^{(m+2)}(1) + J_{\hat{H}}^{(m+2)}(1) \neq 0$, we have shown that $J_{K^{-1}}^{(m+2)}(1) \neq 0$. In \mathcal{G}_{m+2} , $K^{-1} + K = 0$, and since K, K^{-1} are $m+1$ -trivial ($K^{-1} \# K \sim_{m+2} 0 \Rightarrow K^{-1} \# K \sim_{m+1} 0$ and then $K^{-1} \# K \sim_{m+1} 0 \Rightarrow K^{-1} \sim_{m+1} 0$), this implies $J_{K^{-1}}^{(m+2)}(1) = -J_K^{(m+2)}(1) \neq 0$, establishing the key property of the inductive statement.

We must also show that the graph of K has no cycles and has valences less than or equal to 2, and that each edge corresponds to a finger move. When one forms K from H the handle pattern is the same.

To see that the two graph properties are the same, notice K was constructed with one of the two following moves on the graph. The first one is to take a special edge, delete it, and then connect two added vertices to the endpoints of the deleted edge. The second is to delete two edges, add two vertices, and add four edges between the added vertices. The change is as diagrammed in figure 21 and preserves the properties we want. The fact that the edges correspond to finger moves follows since in the added surface T , each added torus has a handle which links a handle of G once in a single clasp. Finally we must exhibit an example for the case $n = 4$. The only possible problem with the above induction would be if when forming H as pictured in figure 22, $J_H^{(2)}(1) = 0$. However, using Knotscape to calculate the Jones Polynomial, we see $J_H(t) = t^{-4} - 2t^{-3} + 3t^{-2} - 4t^{-1} + 5 - 4t + 3t^2 - 2t^3 + t^4$, and one may calculate $J_H^{(2)}(1) = 12 \neq 0$.

□

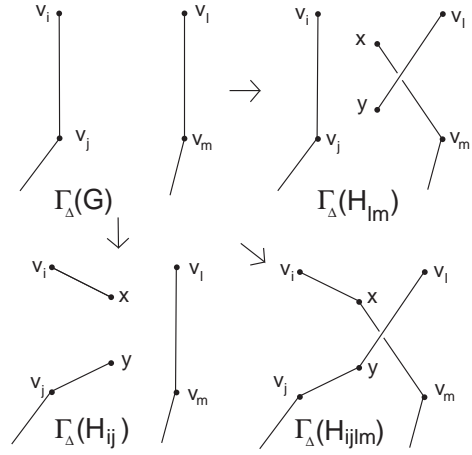


Figure 21: The three possible moves in the construction of H .

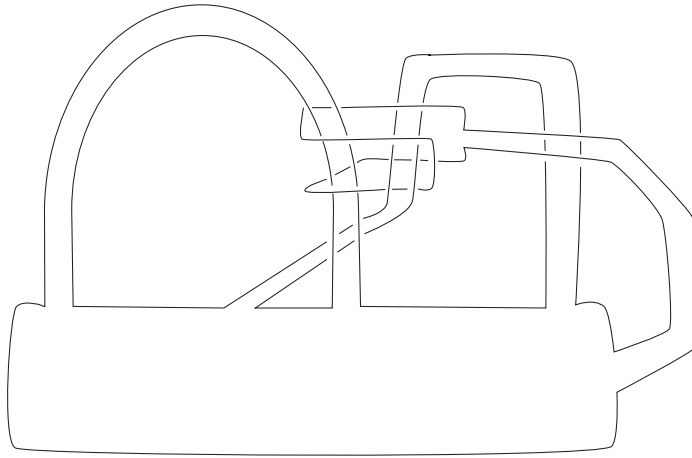


Figure 22: In order for the induction to get started $J^{(2)}(1)$ should not be zero on the above knot.

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